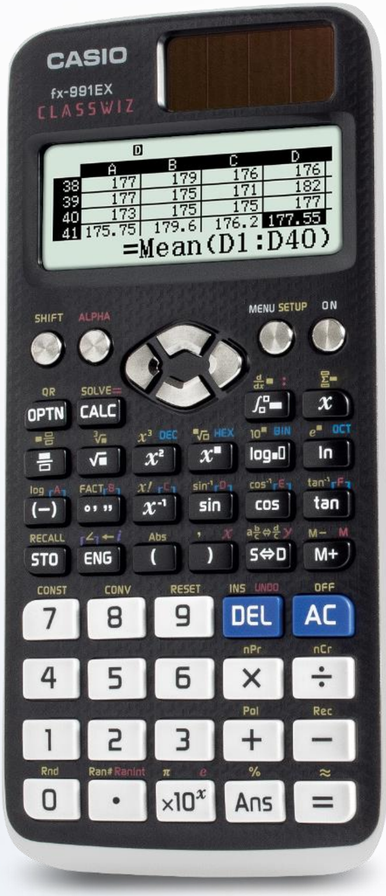


$$\frac{d}{dx}\left(\sqrt[x]{x}\right)\Big|_{x=x}=0$$

$x=$ 2.718281831

$L-R=$ 0

CLASSWIZ
MAKES MATHEMATICS
AND SCIENCE FUN

by Bjørn Bjørneng

CASIO

NOTES

PROLOGUE

Norwegian students are very happy solving problems in science and mathematics with their ClassWiz calculators.

Mathematics is beautiful.

Mathematics gives a person a possibility of understanding and solving problems of different qualities. Mastering is an important key to the adventure of mathematics and science and gives a student better self-confidence and eagerness to continue. Without mastering, a student can feel mathematics and science as only problems with no satisfaction and then lose interest - and that is a pity!

A small calculator, like the ClassWiz fx-991EX, increases the possibility of mastering. It is important that the student (and the teacher) understands and is active.

Exploring different ways of solving problems with this calculator is fun. I have been using calculators for more than 40 years and still being surprised of new possibilities. Some of which I'd like to share with teachers and students using Casio calculators.

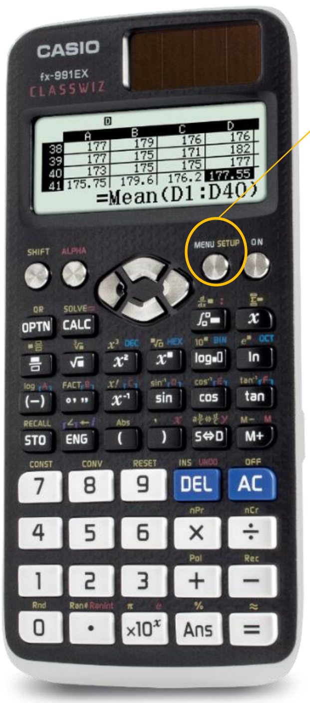
Hopefully this small booklet will give you some ideas and give a little inspiration to further investigations. Thanks to Pepe Palovaara for help and advices.

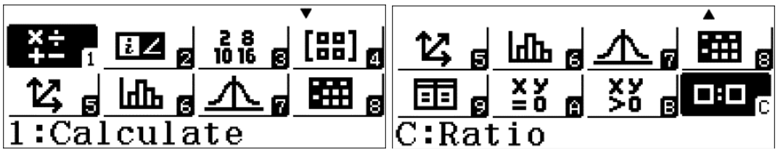
In Bergen, the 24th of September 2018

Bjørn Bjørneng

CONTENT

CHAPTER 1: APPLICATIONS





Click the MENU key to enter the main menu at all times. Select the correct app by using short cut keys 1 - 9 and (−) °′″ x^{-1} or navigate with ▲ ◄ ▼ ► and use = to select app.

1: Calculate: All kinds of calculations and the equation solver
2: Complex: Complex numbers a+bi or polar form
3: Base N: For numbers with base 2, 8, 16 and 10
4: Matrix: Matrices and determinants
5: Vector: Vector calculations for dimensions two and three
6: Statistics: Statistics with regressions
7: Distribution: Normal, binomial and Poisson distributions
8: Spreadsheet: Calculations with cells, rows and columns
9: Table: Function value table for 1 or 2 functions
A: Equation/Function: Polynomials 2 - 6 unknowns of 2nd - 6th degree
B: Inequality: Polynomial functions inequality solver
C: Ratio: Solving A:B = x:C or A:B=D:x

1.1 CALCULATE APP (MENU 1)

You can find answers to all problems at the end of this booklet.

Enter Calculate app (MENU 1) and SETUP (SHIFT MENU) to explore calculator settings.

Input/Output

```
1:Input/Output
2:Angle Unit
3:Number Format
4:Engineer Symbol
```

```
1:MathI/MathO
2:MathI/DecimalO
3:LineI/LineO
4:LineI/DecimalO
```

1: The screen is mathematics all the time (Natural Display).
2: Mathematics in and the result is decimal.
3: Mathematics in and the result is in linear form.
4: Linear in and decimal out, with small font 6 lines can be shown simultaneously.

Example 1: MathI/MathO

Resistances in parallell

$$\frac{1}{\frac{1}{40}+\frac{1}{60}}$$

24

Trigonometry

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

60

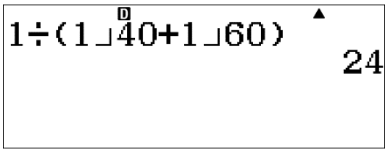
Integral

$$\int_1^{e^2} x\ln(x)dx$$

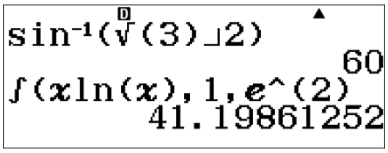
41.19861252

Example 2: LineI/LineO

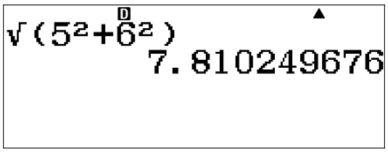
Linear way to show fractions



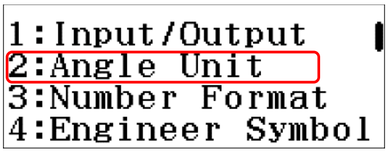
Trigonometry and integrals

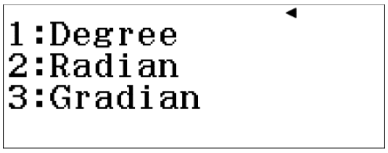


Key x^2 enables upper index



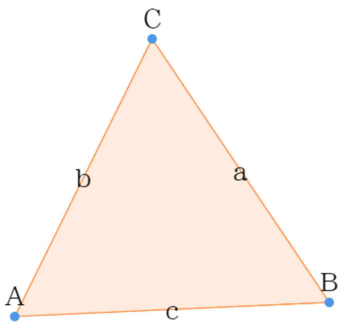
Angle Unit





Degree = full circle $360°$
Radian = full circle 2π
Gradian = full circle 400 gon

Three exercises: (answers on the last page)

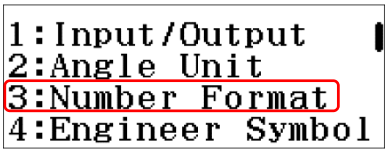


Problem 1: Given $\angle A = 65°$, $b = 10$ and $c = 12$.
Find a, $\angle B$ and $\angle C$.

Problem 2: Given $a = 10$, $b = 8$ and $c = 12$.
Find the angles of the triangle

Problem 3: Given $\angle A = 50°$, $b = 10$ and $a = 8$.
Find $\angle B$, $\angle C$ and c (two solutions).

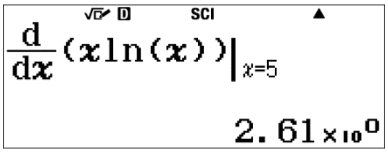
Number Format



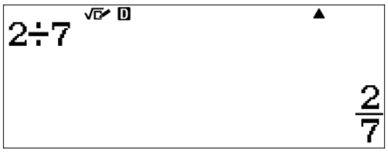
Fix 3 gives answers with 3 decimals, e.g. $20 \div 7 = 2.857$
Sci 3 gives answers with $3\,digits \cdot 10^{power}$, e.g.
$20000 \div 7 = 2.86 \cdot 10^3$

Example 3: Different number formats in calculations.

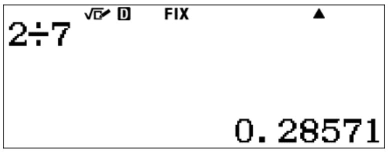
Sci 3



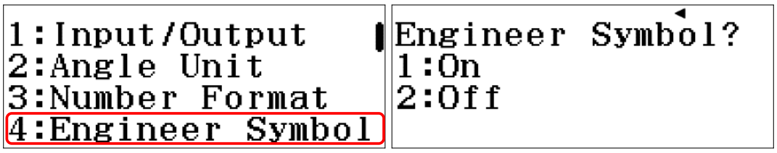
Norm 1



Fix 5 (S⇔D to change to decimal)



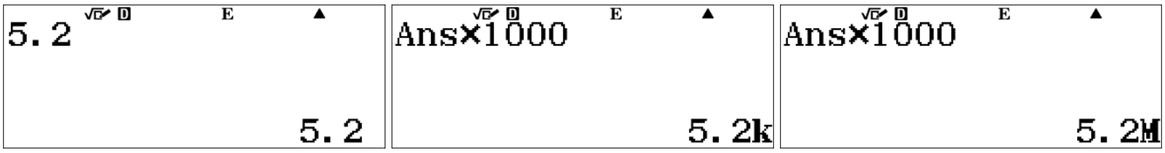
Engineer symbols (k, M, m, µ, etc.)



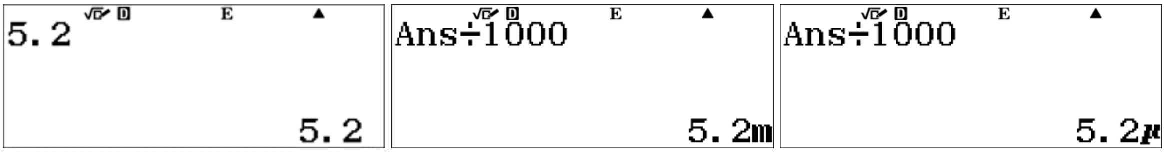
Capital E is shown on the top of the calculator screen to indicate Engineer mode. Return to Engineer Symbol menu to switch it off.

$k=10^3$	$M=10^6$	$G=10^9$	$T=10^{12}$	$P=10^{15}$
$m(milli)=10^{-3}$	$\mu(micro)=10^{-6}$	$n(nano)=10^{-9}$	$p(pico)=10^{-12}$	$f\,(femto)=10^{-15}$

Example 4: 5 • 2 = × 1 0 0 0 = × 1 0 0 0 =



Example 5: 5 • 2 = ÷ 1 0 0 0 = ÷ 1 0 0 0 =



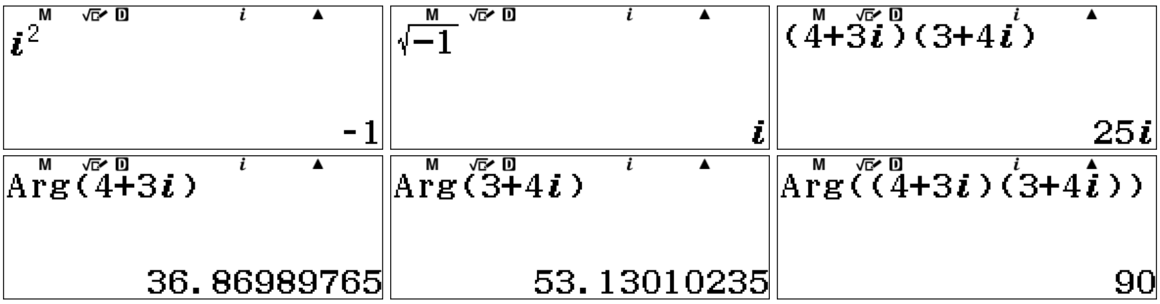
1.2 Complex App (MENU 2)

Complex number calculations with $i=\sqrt{-1}$ and $i^2=-1$. To input i in calculations use keys ALPHA ENG
A complex number Z has a real part a and an imaginary part b and we may write it as $Z=a+bi$.

Example 6: An electric circuit with an inductor, condencator and resistance has got an impedance of $Z=28+10i$. What is the $|Z|$ and the angle of phase?

SHIFT (2 8 + 1 0 ENG = S⇔D OPTN 1 2 8 + 1 0 ENG) =

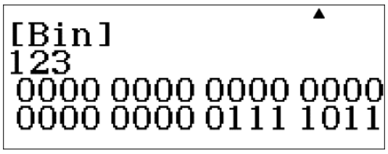
Example 7: Basic tasks with complex numbers.

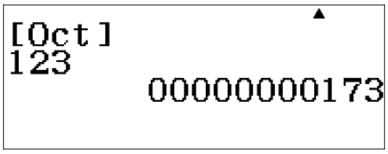


1.3 Base N app (MENU 3)

For calculations using different bases. Use keys x^2 $x^\blacksquare$ $\log_\blacksquare\square$ ln to change the base.

- Dec = Decimal system with base 10
- Hex = hexagonal with base 16
- Oct = octagonal with base 8
- Bin = binary system with base 2



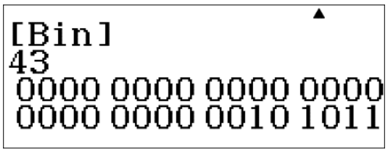


$$1\cdot 20^2+2\cdot 10^1+3\cdot 10^0=1\cdot 2^6+1\cdot 2^5+1\cdot 2^4+1\cdot 2^3+0\cdot 2^2+1\cdot 2^1+1\cdot 2^0=1\cdot 8^2+7\cdot 8^1+3\cdot 8^0$$

Example 8: Binary numbers.

Dec	1	2	3	4	5	6	7	8	9
Bin	1	10	11	100	101	110	111	1000	1001

In binary system, the rules are easy: $1+0=1$, $1+1=10$, $1\cdot 0=0$ and $1\cdot 1=1$. For example $43_{10}=1_{10}+2_{10}+8_{10}+32_{10}=1_2+10_2+1000_2+100000_2=101011_2$. The same transformation with ClassWiz fx-991EX: 4 3 = $\log_\blacksquare\square$



Try first without and then with the ClassWiz:

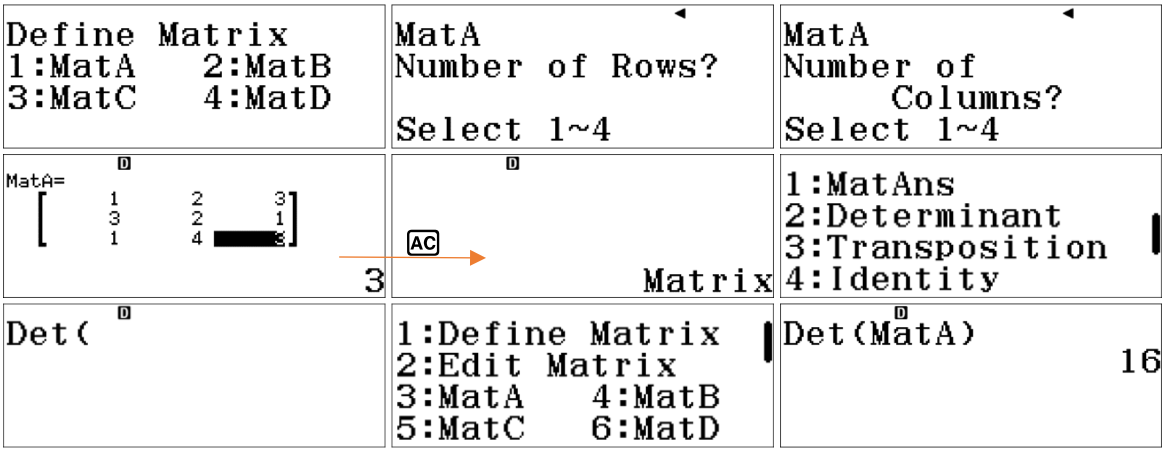
Problem 4: Write 249_{10} as a binary number.

Problem 5: Write the binary 1001111101_2 using decimal system.

Problem 6: Multiply $12_{10}\cdot 13_{10}=156_{10}$ as binary numbers ie. with base 2 and check the answer.

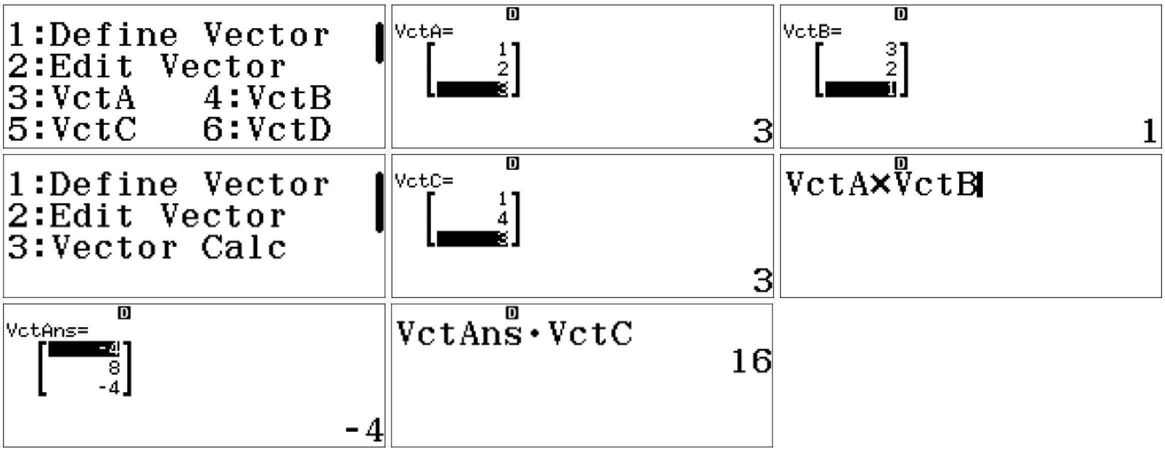
1.4 MATRIX APP (MENU 4)

Example 9: 3×3-matrix with determinant. After inputting the values for matrix A, click AC to change into matrix calculation mode. Use the key OPTN to choose, edit or add matrices.



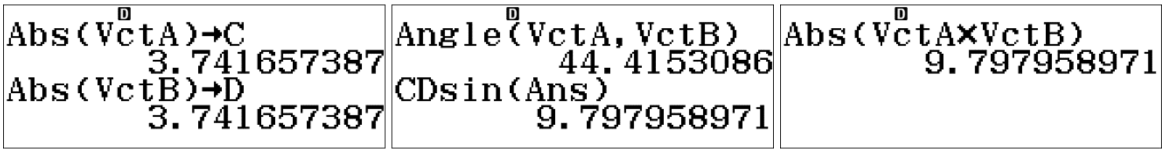
1.5 VECTORS APP (MENU 5)

Example 10: Given vectors $A=[1 \quad 2 \quad 3]$, $B=[3 \quad 2 \quad 1]$ and $C=[1 \quad 4 \quad 3]$ calculate $A \times B$ and $(A \times B) \cdot C$.

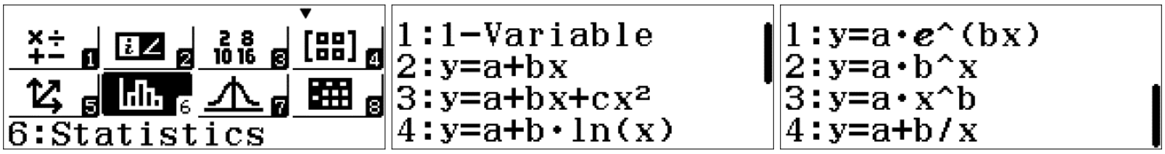


Vector calculation $(A \times B) \cdot C$ in **example 10** has the same result as $Det(MatA) = 16$ in **example 9**. This represents the *volume of the oblique prism* formed by the vectors A, B and C.

Example 11: How to calculate the area of a parallellogram formed by the vectors A and B? The same result can also be calculated by using rules of analytic geometry as shown in the 3rd screen shot. Let's use the vectors A and B from the **example 10**.



1.6 Statistics app (MENU 6)

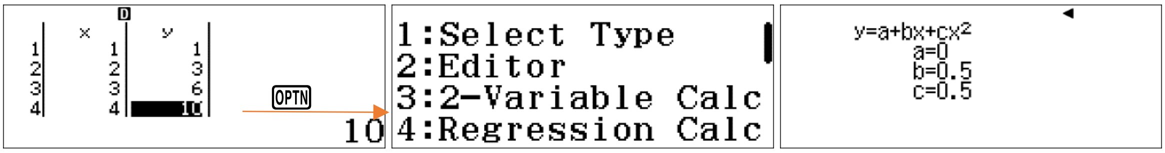


The first 1-Variable is for statitistics analyses. From a list of numbers you find mean, deviation, max, min, and so on. In case you need to use frequences, open SETUP (SHIFT MENU) and scroll down for Statistics to switch on frequences.

Other options give different models of regression between two set of numbers in $list1$ and $list2$. Arrows down ⊙ and up ⊙ can be used to explore these options.

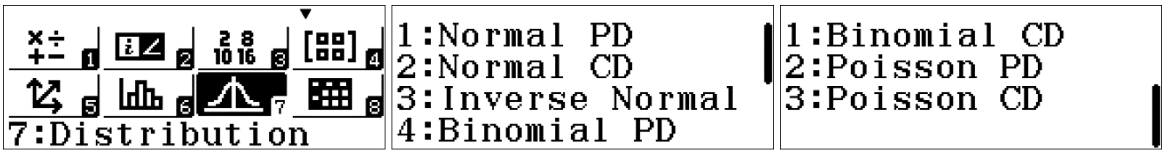
Example 12: Find the sum $S_n = \sum_{x=1}^{n} x = 1+2+3+4+...+n$

n	1	2	3	4
S_n	1	3	6	10



Suggesting the sum to be written as $S_n = \frac{1}{2}n + \frac{1}{2}n^2 = \frac{n(n+1)}{2}$.

1.7 Distribution app (MENU 7)



Permutations (nPr) and hyper geometric distribution ie. combinations (nCr) can be found on keyboard with SHIFT × and SHIFT ÷.

Example 13: In how many ways we can select 4 from a group of 15 when

i) the order is important?

ii) when the the order is unimportant?

Calculations can be done in Calculate app in two different ways.

i)

```
15P4
                32760
```

or alternatively

```
15×14×13×12
                32760
```

ii)

```
15C4
                 1365
```

or alternatively

```
15×14×13×12
    4!
                 1365
```

Example 14: In a class we have 8 boys and 12 girls. We shall select a group of 5 representing the class. What is the probability for selecting x boys and $5-x$ girls when x vary from 0 to 5?

Probability $p($"x boys and $5-x$ girls"$)$ gives us the function $g(x)$. Now, we can utilize the function value table (MENU 9) for $g(x)$ and $f(x)=5-x$, when $x=0,1,2,3,4,5$.

```
g(x)= 8Cx×12C(5−x)
          20C5
```

```
Table Range
 Start:0
 End  :5
 Step :1
```

	x	f(x)	g(x)
1	0	5	0.051
2	1	4	0.2554
3	2	3	[illegible]
4	3	2	0.2383

385⌟969

By investigating the function value table we can conclude

- $p($"Only girls in the group"$) = p(0, f(0)) \approx 5.1\%$
- $p($"Only boys in the group"$) = p(5, f(5)) \approx 0.36\%$
- The most probable selection is 2 boys and 3 girls in the group $p(2, f(2)) \approx 39.7\%$.

Example 15: The probability for a purchase is 80%. There are 20 customers, x customers purchase and the rest $20-x$ don't. Calculate the probabilities for

i) 10, 16 and 20 purchases

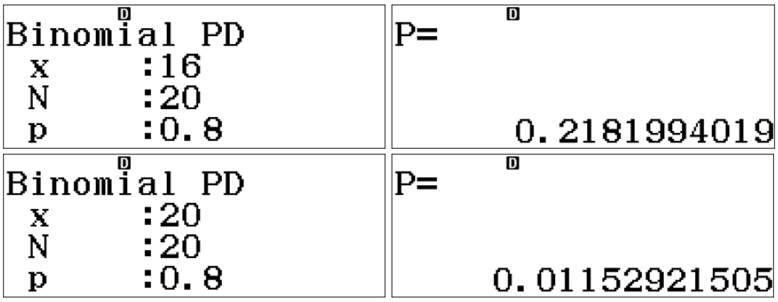
ii) at least 10 purchases

iii) at least 16 purchases

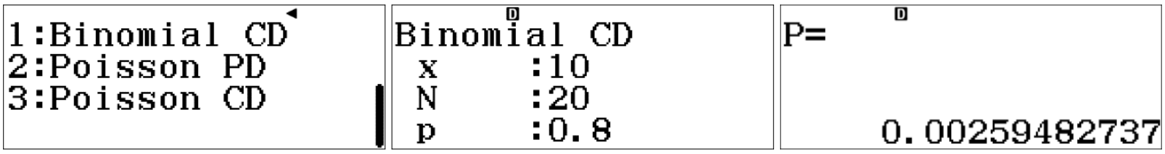
i) Probability for x purchases follows the binomial distribution $p(x) = 20Cx \cdot x^{0.8} \cdot (20-x)^{1-0.8}$ and corresponding values with ClassWiz are

```
Binomial PD
 x    :10
 N    :20
 p    :0.8
```

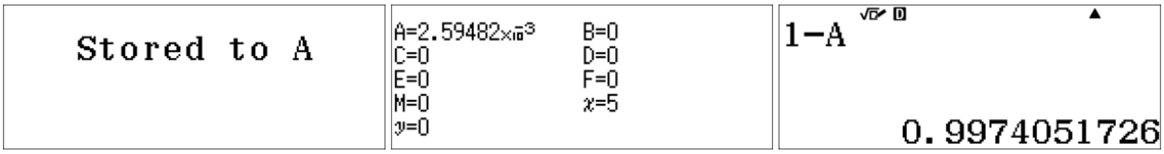
```
P=
        0.0020314137
```



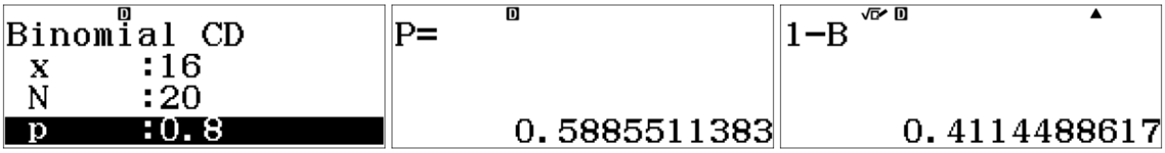
ii) *Binomial PD* stands for *probability density function* giving single values while *Binomial CD* stands for *cumulative distribution function* giving probabilities between boundaries. E.g. for $x = 10$ the Binomial CD means $p(x \leq 10)$ so we have to calculate $p(x > 10) = 1 - p(x \leq 10)$. To change from PD to CD, use keys [OPTN] [1] [2].



To store value into variable A, push [STO] [(−)] and change into Calculate app [MENU] [1] to recall value [SHIFT] [STO] [(−)] for your calculation. Repeat the procedure for the last problem.

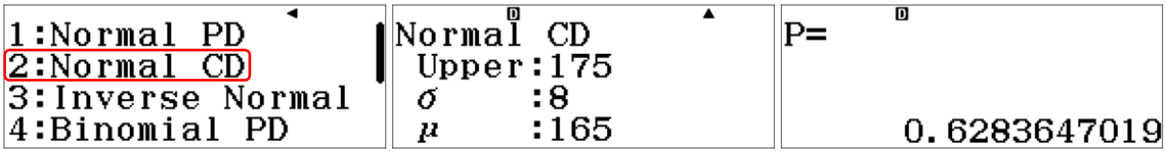


iii) [MENU] [7] [▼] [1] [2] [1] [6] [=] [2] [0] [=] [•] [8] [=] [=] [STO] [°'"] [MENU] [1] [1] [−] [SHIFT] [STO] [°'"] [=]



Example 16: The heights of a group of boys aged 15 are normally distributed with the mean $\mu = 165$ cm and the standard deviation $\sigma = 8.0$ cm. Calculate $p(160 < x < 175)$, where x is the height of a boy (cm).

The Normal CD gives the value of $p(lower < x < upper)$:

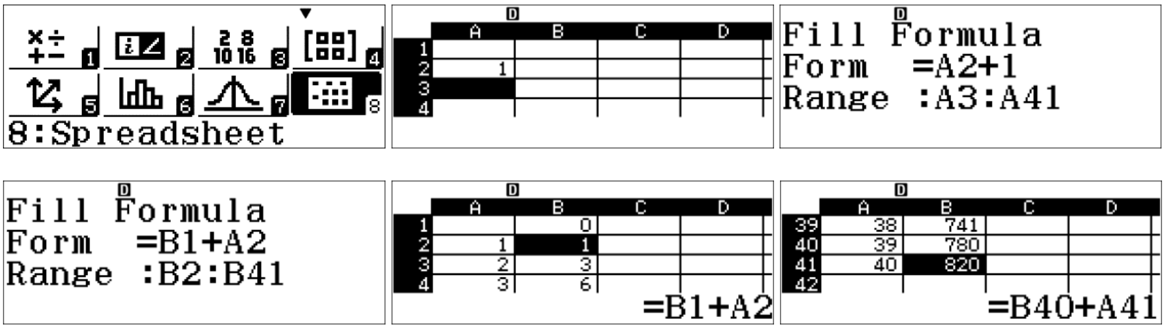


1.8 SPREADSHEET APP (MENU 8)

The **fx-991EX** has a spreadsheet application. The number of rows and columns are limited up to free memory, but we can still have some fun exploring e.g. sequences!

Example 17: A teacher with a Monday morning headache asked his class, at age of 8, to find the sum $1+2+3+...+99+100$. Surprisingly, a small boy gave the answer 5050 after only 2 minutes. “How did you do this?” the teacher asked. “Easily: $100+1=101$, $99+2=101$, $98+3=101$ until $51+50=101$ giving the sum to be $50\cdot 101=5050$.” answered the boy. According to the history of mathematics, it is said the boy was Gauss.

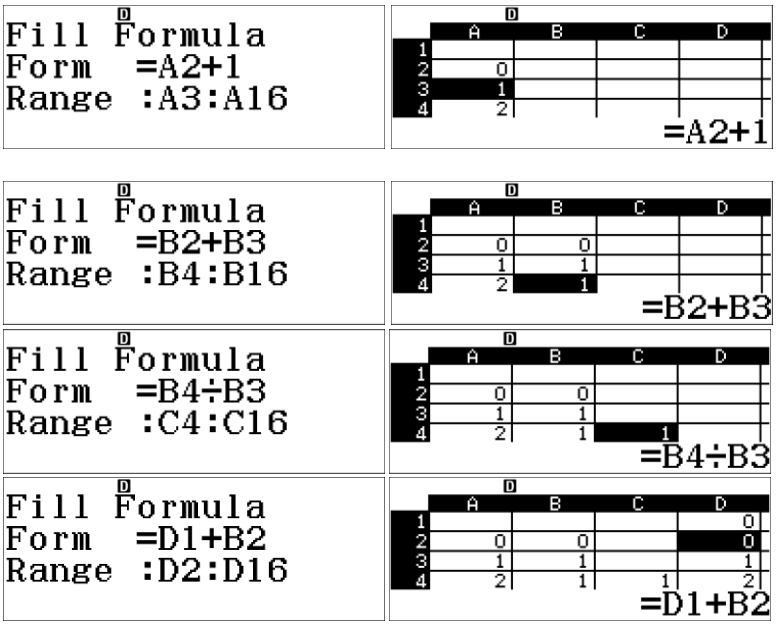
Let’s do something similar with spreadsheet and calculate the sum $1+2+3+...+40$. The method by young Gauss suggests $41\cdot 20=820$. Usually, it’s a good idea to leave row 1 empty, so we start with input $A2=1$. Press key OPTN to fill in formula and set $C1=0$ and the column B to give the sum by using a formula $B2=B1+A2$ (so called *triangle numbers*). You can grab cells by clicking OPTN 2.



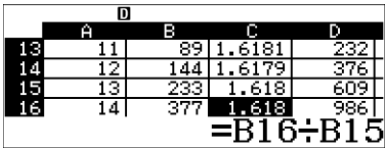
Hint: You may us arrow up ⓐ to move from the first row to the last row.

Example 18: Leave the first row empty and fill in formula for the column A to give $A2=0$, $A3=1,...,\ A16=14$. Start column B with $B2=0, B3=1, B4=B2+B3$, etc. to get the *Fibonacci numbers* till $B16$.

Let’s calculate the ratio of consecutive Fibonacci numbers in column C with $C4=B4\div B3$ all the way to $C18$ and sum of Fiboancci numbers in column D with $D1=0$ and formula $D2=D1+B2$ reaching till $D16$.



Investigating the cells at rows $13-16$, we can see that the ratio of consecutive Fibonacci numbers finds a limit $F_{n+1} \div F_n \to 1.618...$ or $F_n \div F_{n+1} \to 0.618...$ known as *the golden ratio*.



	A	B	C	D
13	11	89	1.6181	232
14	12	144	1.6179	376
15	13	233	1.618	609
16	14	377	1.618	986

=B16÷B15

Comparing the sums in column D to Fibonacci numbers in column B, we can see that $S_n = F_{n+2} - 1$ and deduct a result to get S_n by the known formula for Fibonacci numbers:

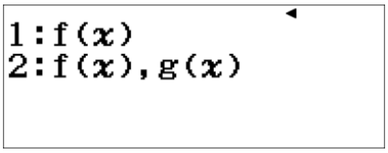
$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

$$S_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right) - 1$$

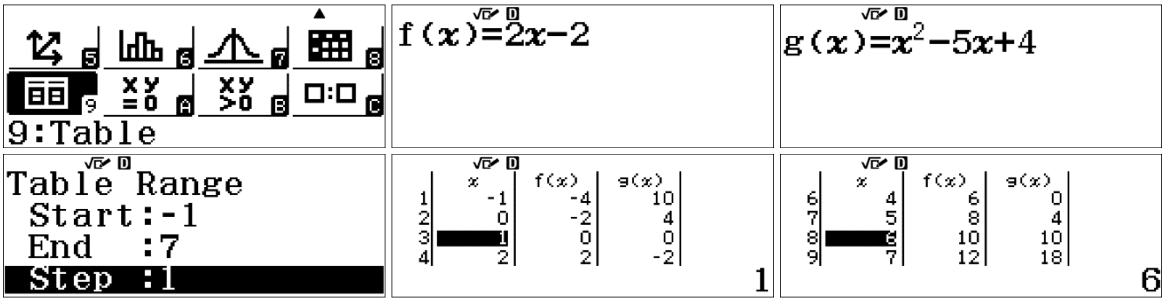
You may explore and investigate new fantastic results in the adventure of mathematics!

1.9 Table app (MENU 9)

With Table app one can create function value tables for one or two functions. After entering the app, settings can be changed with SHIFT MENU and scrolling down to Table.



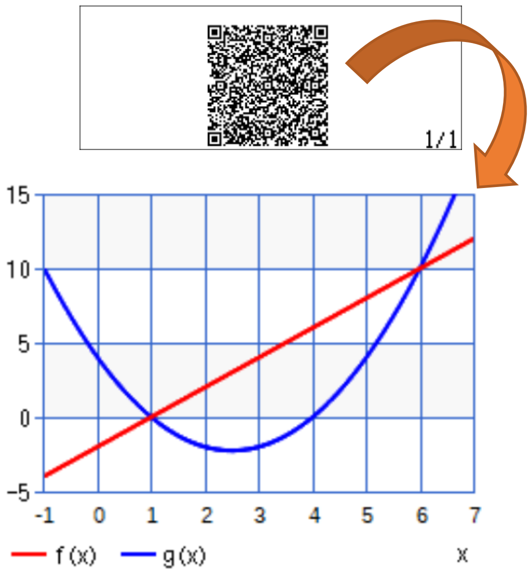
Example 19: Examine when the functions $f(x)=2x-2$ and $g(x)=x^2-5x+4$ have the same value.

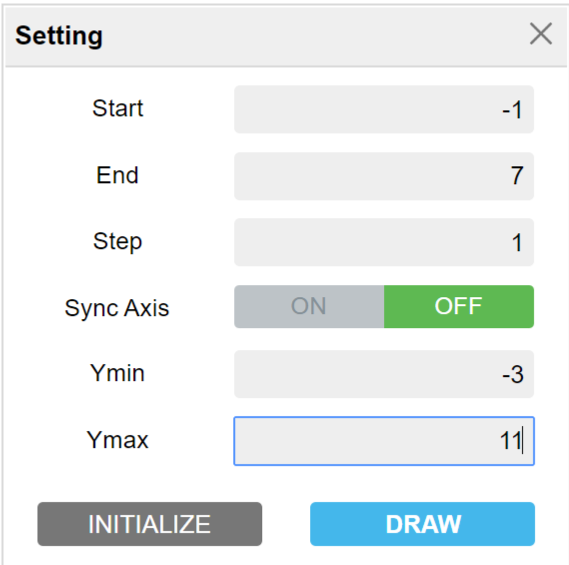


$f(x)=g(x)$ for $x=1$ and $x=6$ according to table. As a line and a parabola cannot have more than two common points, there's no need to search for other entries.

It is possible to obtain the corresponding graph by reading the QR-code (SHIFT OPTN) with a free *Casio EDU+* application (Google Play Shop or Apple AppStore) on a smart device connected to Internet.

If you're using the ClassWiz Emulator on your laptop, you may also click the QR-code appearing in a separate window and access the graph with your web browser.

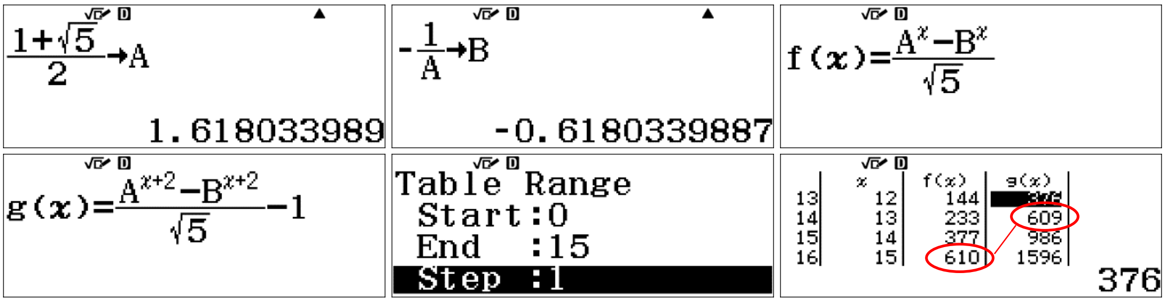




You can change the settings also after seeing the graph in your browser or in Casio EDU+ app. The previous result for common points can be visualized to help understanding the value table.

Example 20: Exploring more the Fibonacci numbers. Let's first save (see the **example 27** to learn the use of memory) two variables in the Calculate app: the ratio of consecutive Fibonacci numbers as a variable A and its inverse (known as *the golden ratio*) with negative sign as a variable B.

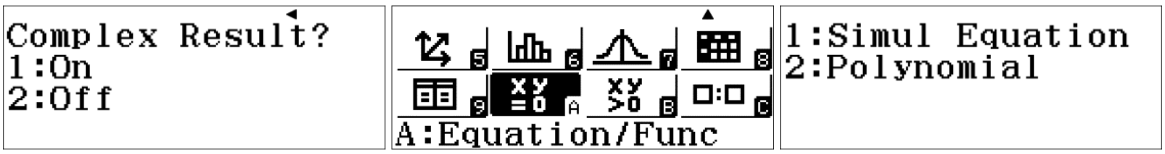
Now, we can define a function f giving us Fibonacci numbers and g calculating corresponding sums in the Table app. We result with the same formula as we found in the **example 18**: the relation between the sum of Fibonacci numbers and the Fibonacci numbers themselves is $S_n = F_{n+2} - 1$.



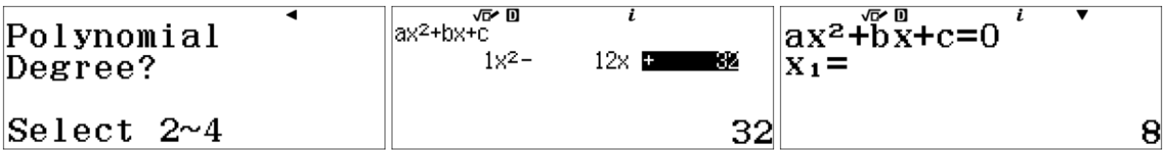
Hint: You can use the variables in calculations by calling them by their name, e.g. ALPHA ∘ for B.

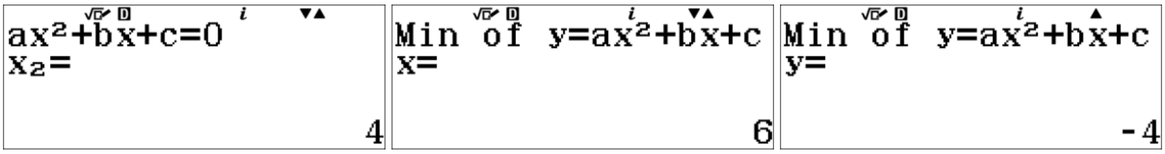
1.10 EQUATION/FUNCTION APP (MENU ⊖)

Application for solving real co-efficient polynomial equations and simultaneous equations. Complex value roots are shown if selected in calculator settings (SHIFT MENU ⏶ ⏶ 1 1) and the indicator ***i*** is shown at the top of the screen. Otherwise calculation results are only shown if real.



Example 21: Solve the equation $f(x) = x^2 - 12x + 32 = 0$ and find the minimum value for $f(x)$.



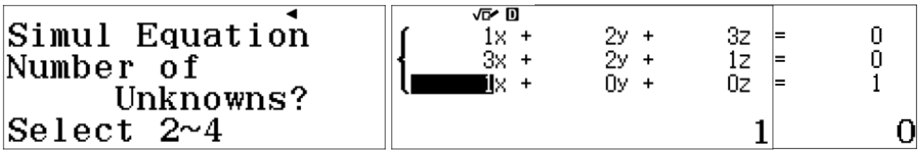


Whenever $a=1$, we find $x_1+x_2=12=-b$ and $x_1 \cdot x_2=32=c$. Please see the corresponding graph and test, if you can solve the equation $x^2-20x+75=0$ using $x_1+x_2=20$ and $x_1 \cdot x_2=75$.

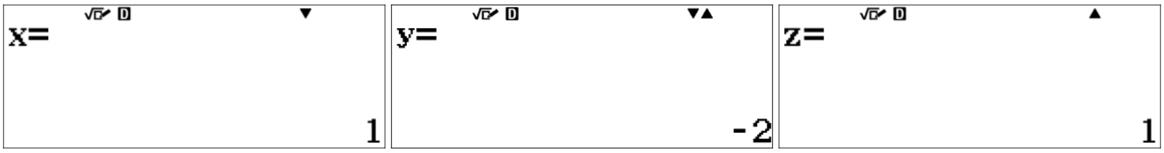
Try also solving these equations without any tools:

a) $x^2-3x-10=0$
b) $5x^2-12x+7=0$ (one solution is $x=1$)
c) $3x^2-10x+8=0$ (one solution is $x=2$)

Example 22: Find one vector $C=\begin{bmatrix} x & y & z \end{bmatrix}$ perpendicular to the vectors $A=\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and $B=\begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$. We can solve this by using simultaneous equations $x+2y+3z=0$, $3x+2y+z=0$ and (because there are 3 variables, but only 2 equations) a dummy $x=1$:



Hint: If a dummy $x=1$ doesn't work, try $y=1$ or $z=1$.

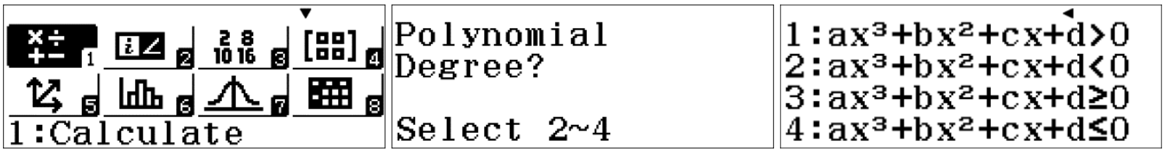


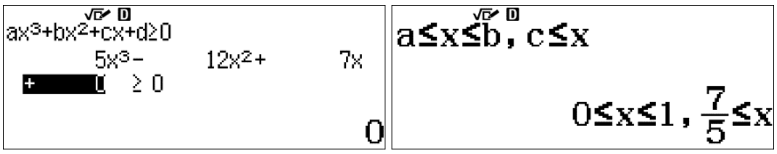
Thus, one vector is $C=\begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$ and another can be $\begin{bmatrix} -4 & 8 & -4 \end{bmatrix}$ given by $A \times B$ or $-4C$.

1.11 Inequality App (MENU)

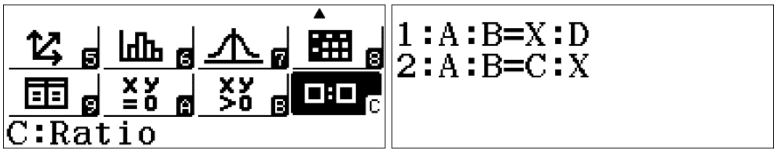
Application for solving real co-efficient polynomial inequations.

Example 23: Solve $5x^3-12x^2+7x \geq 0$.



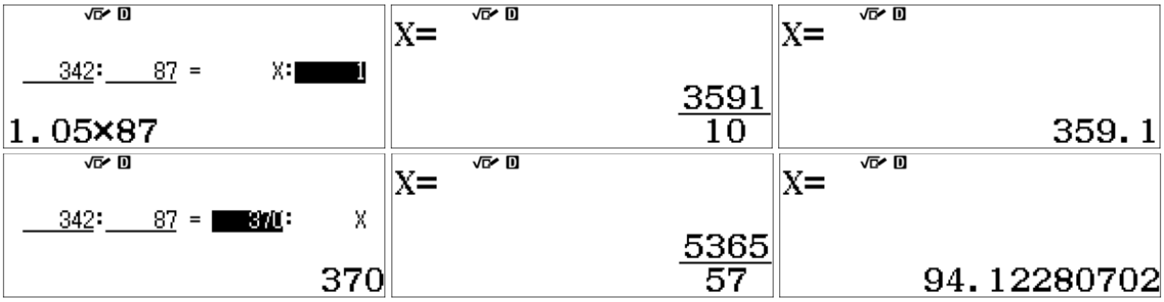


1.12 Ratio app (MENU x^{-1})



For solving basic ratios where 3 out of 4 variables are known and one needs to be solved.

Example 24: Driving 342 km with average speed 87 km/h is possible in the normal traffic. How far could you drive in the same time, if the average speed was be raised by 5% ? How fast should you drive to cover 370 km at the same time?



With 5% increase in average speed the distance covered in the same time would be appr. 359 km while 370 km distance in the same time would require average speed of appr. 94 km/h.

ClassWiz Emulator helps to make materials, plan lessons or to use full strength of the fx-991EX on your Windows computer.

Extra pop-up display, keylog showing key strokes and a one-click access from QR-codes to graphical online visualization service are included.

Free 90 day trial download at https://edu.casio.com >

SOFTWARE / APP

CHAPTER 2: EXAMPLES AND USE OF THE KEYBOARD

Many of the keys have three functions, some two and some only one function. To learn to use keyboard helps to navigate through math tasks, saves time and makes it possible to discover the wonderful world of mathematics with fx-991EX. The 2nd function of the key can be accessed by using the SHIFT key before touching the actual key.

2.1 USING THE x KEY

One of the most commonly used keys is the x key.
With SHIFT it gives the sum function, which is the 2nd function of the key x.

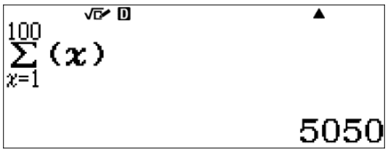
Example 25: Solve the following tasks (Calculate app):

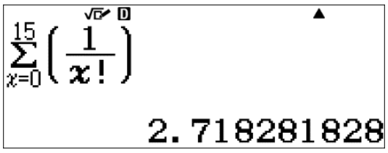
i) Calculate the sum $1+2+...+100$.

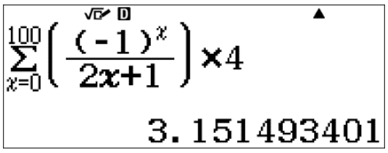
ii) Estimate the Euler number with sum $e \approx \frac{1}{0!}+\frac{1}{1!}+...+\frac{1}{15!}$

iii) Use Leibnitz's method to find π by using series

$$\frac{\pi}{4}=\frac{(-1)^0}{2\cdot 0+1}+\frac{(-1)^1}{2\cdot 1+1}+\frac{(-1)^2}{2\cdot 2+1}+...$$







Note: Euler number is not periodic.

2.2 USING THE CALC KEY

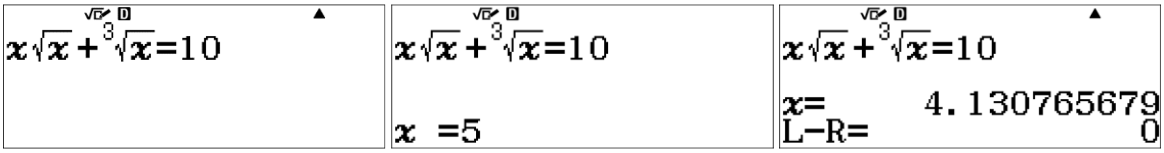
In the Calculate app we can use the equation solver, which is located as the 2nd function of the CALC key. It uses *Newton-Rhapson method* for numerical solving.

Remember these points when typing the equation:

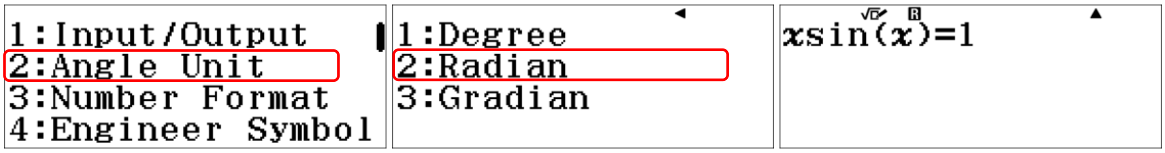
- Use ALPHA CALC for the equal sign =
- Use SHIFT CALC to enter initial value for x
- Use = to solve the equation
- To find several roots, change the initial value of x

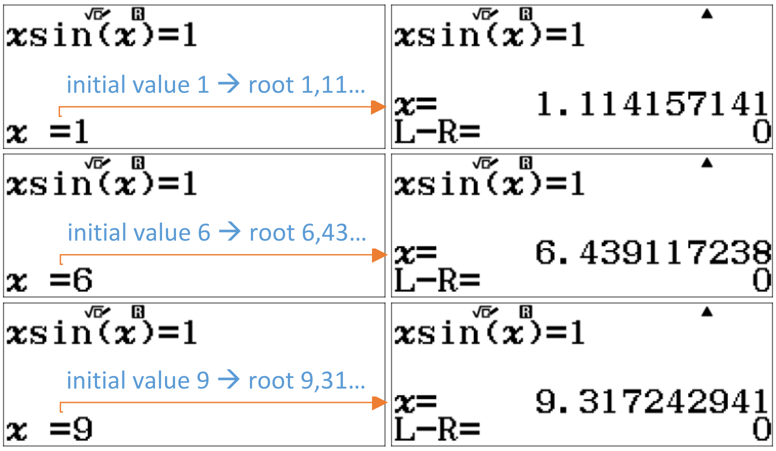
Example 26: Solutions for an equation having only one root and for another equation with several roots. For the trigonometric function the angle unit is changed from degrees to radians. The angle unit indicator on the calculator screen top changes respectively.

[x] [√■] [x] [▶] [+] [SHIFT] [√■] [x] [▶] [ALPHA] [CALC] [1] [0] [SHIFT] [CALC] [5] [=] [=]



[SHIFT] [MENU] [2] [2] [x] [sin] [x] [)] [ALPHA] [CALC] [1] [SHIFT] [CALC] [1] [=] [=] [SHIFT] [CALC] [6] [=] [=] [SHIFT] [CALC] [9] [=] [=]





Important: Whenever solving an equation, pay close attention **not to mix initial values with solutions**!

Under the solution, you'll always find L-R showing the size of an error.

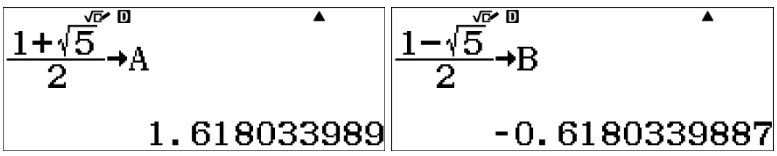
2.3 USING THE [STO] KEY

Using the variable memory in calculations can save a lot of work and make expressions look easier. Click the key [STO] and the key having letter above with red font.

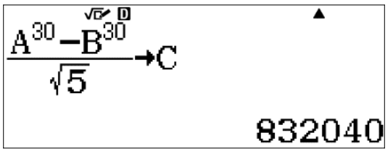
Example 27: For the Fibonacci numbers we'd like

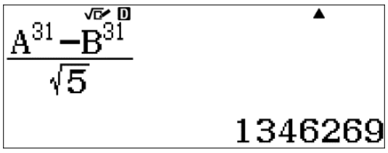
A to have the value $\frac{1+\sqrt{5}}{2}$ and B the value $\frac{1-\sqrt{5}}{2}$.

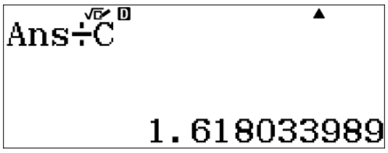
[▭/▭] [1] [+] [√■] [5] [▼] [2] [STO] [(−)] [S⇔D] [▭/▭] [1] [−] [√■] [5] [▼] [2] [STO] [°'"] [S⇔D]



We'd like to check the ratio $\frac{F_{n+1}}{F_n}$ for $n=30$ pushing us to calculate the 30th and the 31st Fibonacci numbers. We can save the 30th Fibonacci number as a variable C and use the previous calculation result variable ***ans*** to finish this calculation. The first screen shot keystroke is shown here:

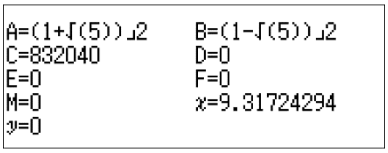






We find the ratio beeing A; this is why the golden ratio is important for Fibonacci numbers.

Note: Anytime with keys SHIFT STO we recall what is stored:

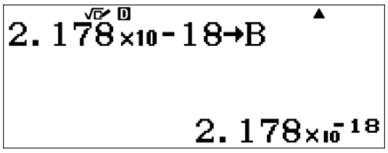


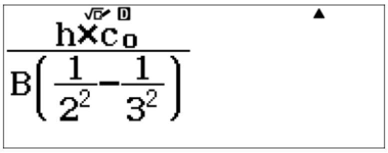
2.3 Using the constants SHIFT 7

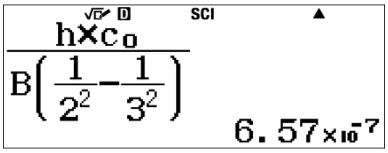
fx-991EX has got 47 constants following CODATA recommendations from 2010. These can be easily added to calculations and combined with own variables.

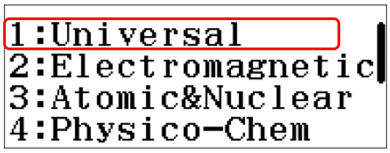
Example 27: Calculate the wave length of H_α.

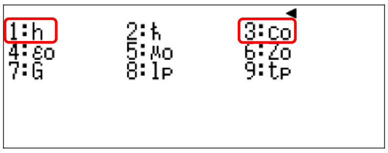
Let's start by defining B as the Bohr's constant. By using it and inbuilt constants, we can calculate the wave length of H_α defined by $\lambda = \dfrac{hc}{B(\frac{1}{2^2}-\frac{1}{3^2})}$.











Note: Calculation utilizes both universal constants and a user defined variable B.

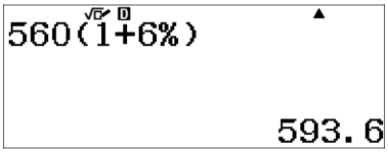
To change into scientific number format, enter the setup and number format SHIFT MENU 3 2 3.

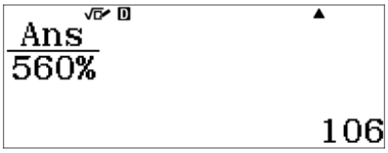
2.4 Using the previous answer variable [Ans]

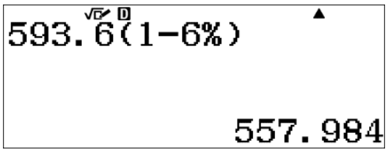
The key [Ans] offers the previous result as a variable which can be used in further calculations. The combination [SHIFT] [Ans] gives the percentage.

If you continue with any of the calculation functions, "Ans" will appear automatically to the next calculation.

Example 28: How much is $560 + 6\%$? How many $\%$ is the result of 560 ? What do you get when you substract 6% from 593.60 ?







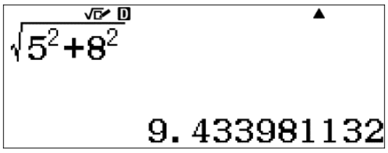
2.5 Using approximation [SHIFT] [=]

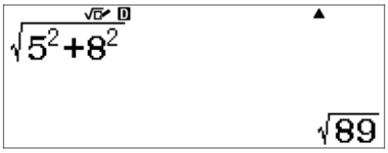
In the MathI/MathO mode calculation results are shown as accurate values with natural display system. However, the approximate value is sometimes needed to e.g. estimate the number. The fastest way to get the result as an approximate value is to use [SHIFT] [=] instead of [=].

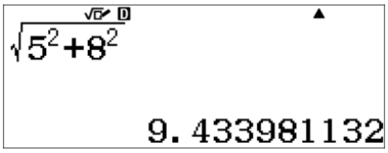
It's also possible to switch between the accurate value and the approximate value with the [S⇔D] key.

Example 29: Calculate the length of the hypotenuse of a triangle with legs 5 and 8.

[√■] [5] [x^2] [+] [8] [x^2] [SHIFT] [=]   or by using the [S⇔D] key: [√■] [5] [x^2] [+] [8] [x^2] [=] [S⇔D]



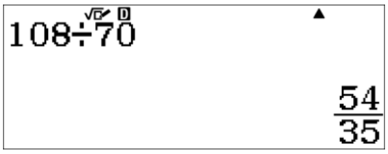


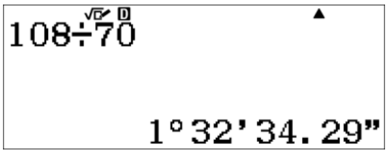


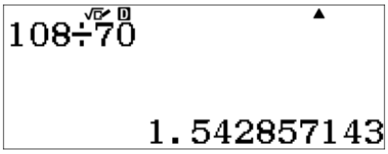
2.6 Time calculations [°'"]

Example 30: What is the time used to drive 108 km with the average speed of 70.0 km/h?

[1] [0] [8] [÷] [7] [0] [=] [°'"] [°'"]



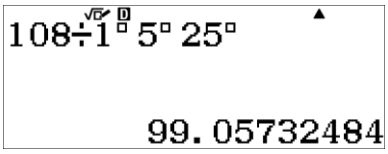




So, the time needed is 1 hour 32 minutes and 34 seconds (1.54 hours).

Example 31: Another driver uses 1 hour 5 minutes and 25 seconds on the same distance. What was the average speed?

[1] [0] [8] [÷] [1] [°'"] [5] [°'"] [2] [5] [°'"] [=] [°'"]



The average speed was appr. 99.1 km/h.

Problem 7: The distance between cities A and B is exactly 100 km. Ann and Peter drive back and forth and both starts in A at 9 o'clock. Ann has the average speed 70.0 km/h both ways, while Peter has the average speed 80.0 km/h from A to B and 60.0 km/h on the way back. Who is back first at A and at what time?

If Ann overtakes Peter at any point, calculate when does this occure and how far from the city B it happens?

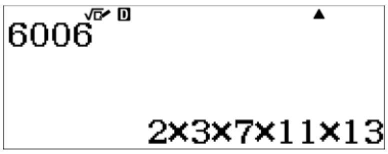
2.7 Factor command [SHIFT] [°'"]

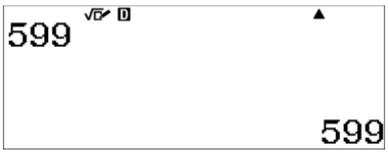
Example 32: Factorize 6006, 599 and 601.

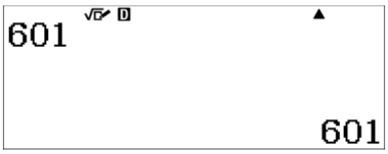
[6] [0] [0] [6] [=] [SHIFT] [°'"]

[5] [9] [9] [=] [SHIFT] [°'"]

[6] [0] [1] [=] [SHIFT] [°'"]





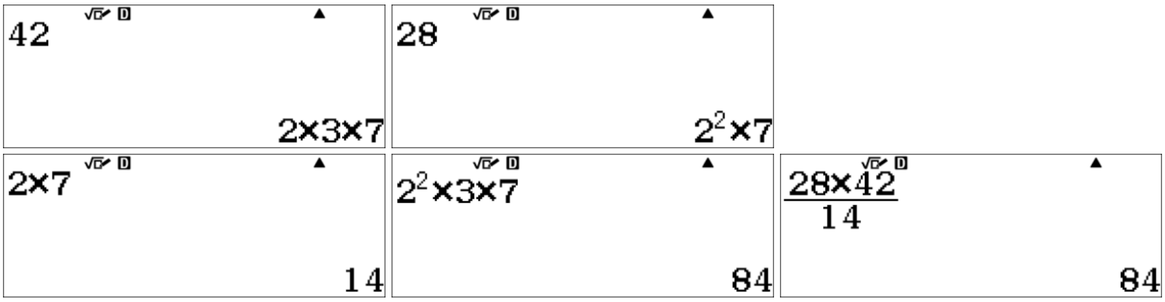


It appears that 599 and 601 are primes and even *prime twins*. Two neighbouring odd numbers both beeing primes we call prime twins, e.g. 5 and 7, 11 and 13, 17 and 19. An even number between two prime twins > 3 is divisible by 6. Try to investigate this with your calculator and prove it!

Example 33: Find the greatest common divisor (GCD) and the lowest common multiple (LCM) for 42 and 28.

For the GCD we need to factorization and then select all the common primes with lowest exponents included and multiply them. For the LCM we take all the existing primes of both numbers and the highest existing exponents of them and multiply.

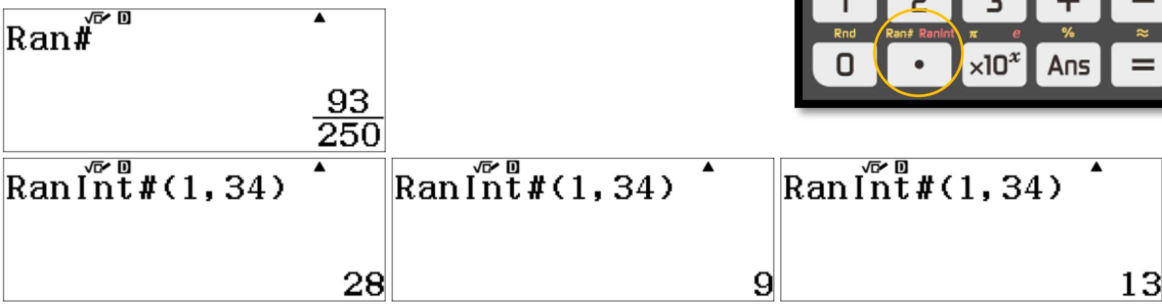
Hint: You can also find the LCM by multiplying the given numbers and dividing the result by GCD.



Thus the $GCD(28,42)=14$ and the $LCM(28,42)=84$.

2.8 Random numbers SHIFT • and ALPHA •

SHIFT • gives the command Ran# for an arbitrary number with three decimals or as a fraction between 0 and 1.
ALPHA • gives the command Ranint(n,m) for an arbitrary integer $x, n \le x \le m$. Touch the = to generate more!



2.9 Fractions

Problem 8: The average of two fractions is defined as $\dfrac{\frac{a}{b}+\frac{c}{d}}{2}$. Find the average of fractions

a) $\frac{1}{4}$ and $\frac{1}{6}$ b) $\frac{1}{8}$ and $\frac{1}{6}$ c) $\frac{3}{8}$ and $\frac{5}{7}$

Try to guess first with these hints before calculating:

$\dfrac{\left(\frac{1}{4}+\frac{1}{6}\right)}{2}$	$\frac{1}{2}\times(6\times1+1\times4)\div(4\times6$	$\dfrac{1}{4.8}$

Think about this problem related to **problem 7**: what is the average speed when at first driven 100 km distance at speed 60 km/h followed by another 100 km with the speed 40 km/h? Can you now solve **problem 7**?

$$\frac{200}{\frac{100}{60}+\frac{100}{40}} \qquad 48$$

CASE STUDY: ARCHIMEDES AND DEFINING π

With a small calculator like ClassWiz, we can follow the working and ideas of the great ancient mathematics, e.g. how Archimedes found π .

$$\frac{223}{71} < \pi < \frac{22}{7}$$

Archimedes of Syracuse

He lived from $287-212$ BC and was a Greek mathematician, physicist, engineer, inventor and astronomer. Although few details of his life are known, he is regarded as one of the leading scientists in classical antiquity and generally considered the greatest mathematician of antiquity and one of the greatest of all

(picture: http://www.kidsmathgamesonline.com/pictures/mathematicians/archimedes.html)

$\frac{223}{71}$ 3.14084507	<	π 3.141592654	<	$\frac{22}{7}$ 3.142857143

Archimedes anticipated modern calculus and analysis by applying concepts of infinitesimals and the method of exhaustion to derive and rigorously prove a range of geometrical theorems, including the area of a circle, the surface area and volume of a sphere, and the area under a parabola and he derived a very accurate approximation of π finding

$$\frac{223}{71} < \pi < \frac{22}{7} \Leftrightarrow 3.1408 < \pi < 3.1428;\ \bar{\pi} = 3{,}1418$$

Archimedes' tools

1. He knew how to make fractions as good approximations for roots. E.g. how to estimate $\sqrt{70}$?

   The first approximation of $\sqrt{70} = 8.5 = \dfrac{17}{2}$ as an average of neighbouring squared integer roots $8 = \sqrt{64} < \sqrt{70} < \sqrt{81} = 9$. The next approximation will be the average of the first approximation and 70 devided by the first approximation giving

   $$\frac{\frac{17}{2}+\frac{70\times 2}{17}}{2} = \frac{569}{68} \qquad \frac{\frac{17}{2}+\frac{70\times 2}{17}}{2} = 8.367647059 \qquad \text{Ans}^2 = 70.0175173$$

   and by repeating this procedure he could finally conclude

   $$\frac{\frac{569}{68}+\frac{70\times 68}{569}}{2} = 8.366600331 \qquad \text{Ans}^2 = 70.0000011 \qquad \sqrt{70} = 8.366600265$$

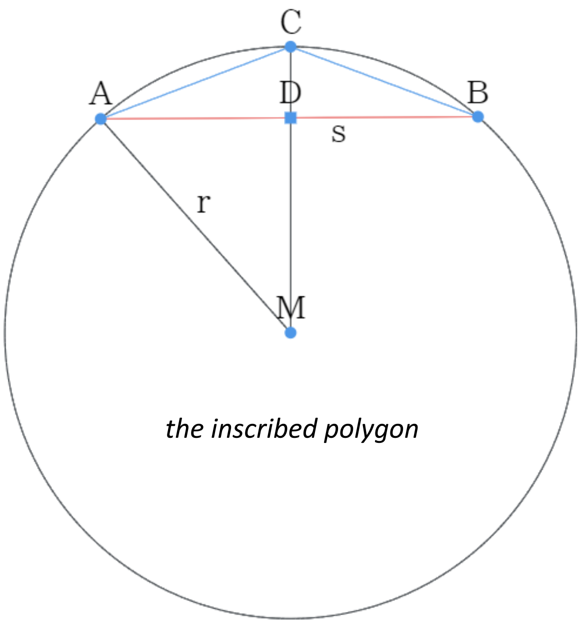
   This must be good enough as the approximation has 6 correct decimals already as we can check with the calculator shown in the last screen shot.

2. He knew the Pythagorean theorem of right-angled triangles.

Compared to some old approximations for π, Archimedes' method is very impressive. E.g. in the Bible $\pi \approx 3$ and from the ancient Egypt $\pi \approx \dfrac{22}{7}$.

Archimedes' method

In a circle with radius $r = 1$, Archimedes inscribed a regular polygon with n sides with length s giving the circumference $n \cdot s$. He started with a known polygon and continued with a recursion to find the length of the side when the number of sides is doubled. Then he did the same with a circumscribed polygon aiming for the average of these two circumferences.



the inscribed polygon

Step 1: We start with the inscribed polygon. Let the M being the centre of circle, $AB = s_n = s$ is the side length of a regular polygon with n sides and BC will be the length of the next polygon with $2n$ sides. The recursion will be to express BC as a function of $AB = s$.

$$AD = \frac{s}{2};\ MD = \sqrt{1-\left(\frac{s}{2}\right)^2} \Rightarrow DC = 1-\sqrt{1-\left(\frac{s}{2}\right)^2}$$

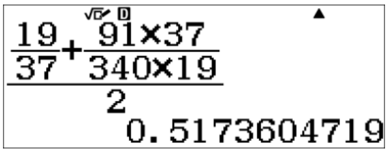
$$CB = \sqrt{DC^2 + DB^2} = \sqrt{\left(1-\sqrt{1-\left(\frac{s}{2}\right)^2}\right)^2 + \left(\frac{s}{2}\right)^2} = \sqrt{2-\sqrt{4-s^2}}$$

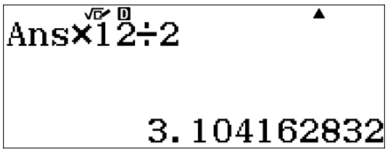
We start with $n = 6$ making the side length equalling the radius, ie. $s_6 = 1$ giving $s_{12} = \sqrt{2-\sqrt{3}}$ as proved for CB above. By knowing roughly that $\sqrt{3} \approx 1.7$ and with the Archimedes' method, we get

$$\sqrt{3} \approx \frac{17}{10} \Rightarrow \sqrt{3} \approx \frac{\frac{17}{10}+\frac{30}{17}}{2} = \frac{589}{340}$$

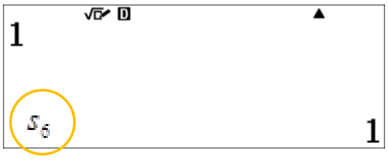
$$\sqrt{2-\sqrt{3}} \approx \sqrt{2-\frac{589}{340}} = \sqrt{\frac{91}{340}} \approx \frac{\frac{19}{2}}{\frac{37}{2}} = \frac{19}{37}\sqrt{\frac{91}{340}} \approx \frac{\frac{19}{37}+\frac{91\cdot 37}{340\cdot 19}}{2}$$

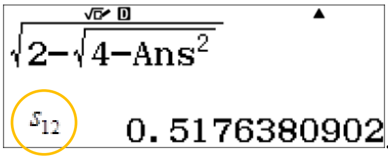
Thus we can calculate our first estimation of π by using regular polygon with 12 sides:





We now take over with our ClassWiz fx-991EX by using the previous answer Ans to make the same recursion as Archimedes, but in more modern way. Recursion starts with a regular hexagon with side length of 1. Each step of the recursion $\sqrt{2-\sqrt{4-Ans^2}}$ doubles the number of sides in our regular polygon and results the corresponding side length. The recursion can quickly be repeated by simply touching the = key: 1 = √ 2 − √ 4 − Ans x^2 = = = = = =





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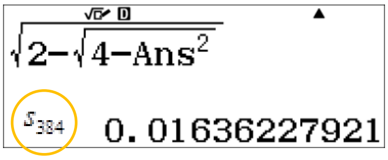
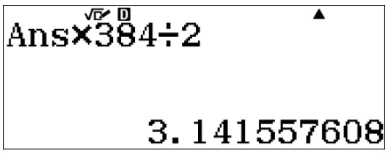
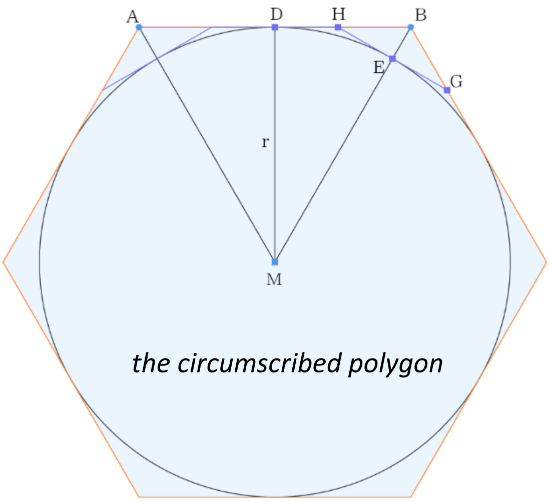


Now we get more accurate estimation of π by using a regular polygon with 384 sides having the side length $s_{384} \approx 0.01636227921$. We know this step 1 value is a little bit too small, because we used the inscribed polygon





the circumscribed polygon

Step 2: We continue with the circumscribed polygon. $AB = s_n = s$ is the size length for regular hexagon. Let's mark $DH = HE = EG = x$ and $HG = s_{2n} = 2x$. Let's find $2x$ as a function of s.

$$DB = \frac{s}{2};\ HB = \frac{s}{2} - x$$

$$MB = \sqrt{1 + \frac{s^2}{4}};\ EB = \sqrt{1 + \frac{s^2}{4}} - 1$$

Because $\Delta\, HEB$ is rightangled, we get

$$HE^2 = HB^2 - EB^2$$

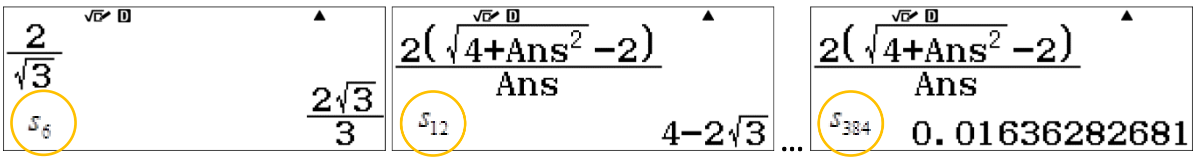
$$x^2 = \left(\frac{s}{2} - x\right)^2 - \left(\sqrt{1 + \frac{s^2}{4}} - 1\right)^2$$

$$= \frac{s^2}{4} - sx + x^2 - 1 - \frac{s^2}{4} + 2\sqrt{1 + \frac{s^2}{4}} - 1 \Rightarrow sx = 2\sqrt{1 + \frac{s^2}{4}} - 2 \Rightarrow 2x = \frac{2\left(\sqrt{4 + s^2} - 2\right)}{s}$$

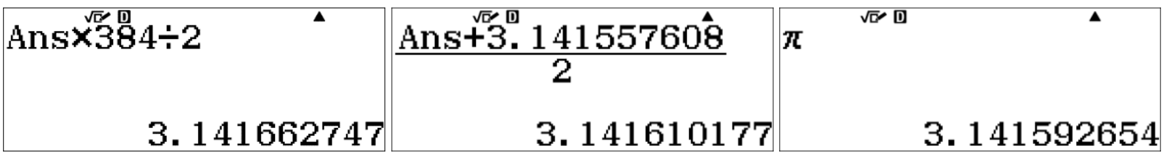
This gives us the recursion formula for calculating the side length of a circumscribed polygon whenever the number of sides is doubled

$$s_{2n} = \frac{2\left(\sqrt{4 + s_n^2} - 2\right)}{s_n}$$

We know that Arcimedes used this recursion. He could have started with a polygon with 4 or 6 sides. We may use a polygon with 6 sides having the side length of $s_6 = \frac{2}{\sqrt{3}}$ and repeat the recursion: 2 ▤ √■ 3 = 2 ▤ Ans ▲ (√■ 4 + Ans x² ▶ − 2) = = = = = =



This **step 2** gives us a little bit too big estimation as we used the circumscribed polygon. By taking the average of **steps 1** and **2** solutions, we get very nice approximation for π accurate to 4 decimals.



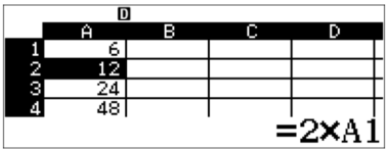
We challenge the reader to start with regular polygons with 4 sides. For the inscribed polygon $s_4 = \sqrt{2}$ and for the circumscribed polygon $s_4 = 2$. The recursions will be the same.

Note: The ClassWiz fx-991EX has also a spreadsheet application, which can be used to solve the same exercise to estimate π. We may list down

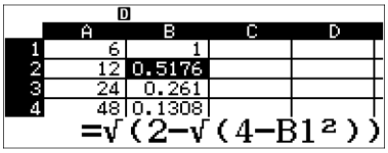
- the number of inscribed polygon sides to column A starting with $A1 = 6$
- the lengths of the corresponding sides to column B by using the recursion
- estimations for π in the column C as the circumferences of the inscribed n-polygons
- the lengths of the circumscribed n-polygon sides to column D starting with $D1 = \dfrac{2}{\sqrt{3}}$
- estimations for π in the column E as the circumferences of the circumscribed n-polygons

Step 1 with spreadsheet: For the inscribed polygon the input is as follows.

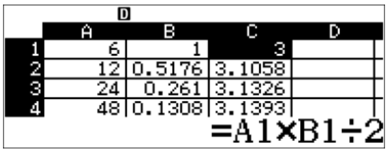
6 = OPTN 1 2 × OPTN 2 ▲ = = ▶ ▶ ▶ ▶ ▶ ▶ DEL 1 6 = =



1 = OPTN 1 √ 2 − √ 4 − OPTN 2 ▲ = x^2)) = ▶ ▶ ▶ ▶ ▶ ▶ ▶ DEL 1 6 = =

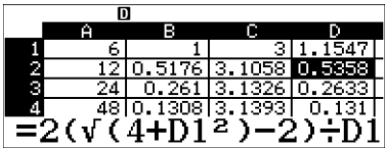


OPTN 1 OPTN 2 ◀ ◀ = × OPTN 2 ◀ = ÷ 2 = ▶ ▶ ▶ ▶ ▶ ▶ 6

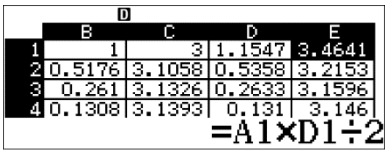


Step 2 with spreadsheet: For the circumscribed polygon we may type

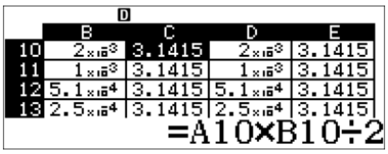
2 ÷ √ 3 = OPTN 1 2 (√ 4 + OPTN 2 ▲ = x^2) − 2) ÷ OPTN 2 ▲ = = ▶ ▶ ▶ ▶ ▶ ▶ DEL 1 6 = =

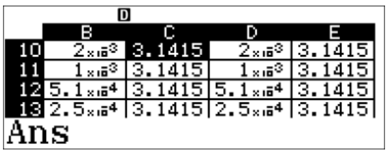


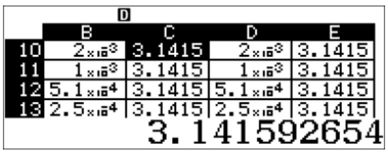
OPTN 1 OPTN 2 ◀ ◀ ◀ ◀ = × OPTN 2 ◀ = ÷ 2 = ▶ ▶ ▶ ▶ ▶ ▶ 6 = =



To change the calculation formulas into pure values, choose the cell and click the keys Ans = ▲



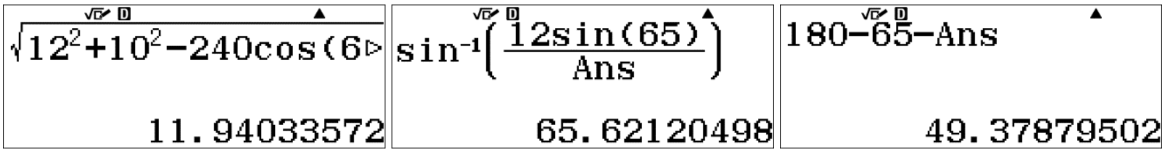




The average of cells $C16$ and $E16$ gives $\pi \approx 3.141600738$.

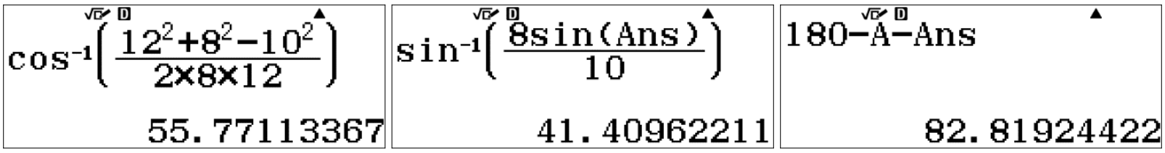
Chapter 3: Answers to problems

Problem 1:



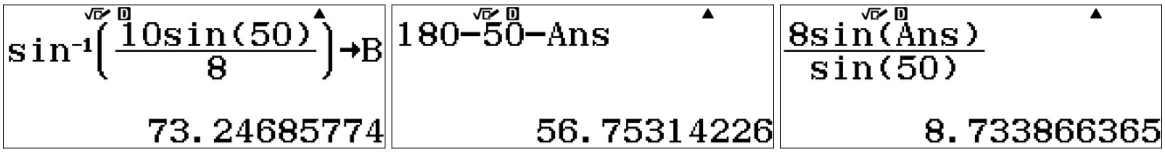
$a \approx 11.9$, $\angle C \approx 65.6°$ and $\angle B \approx 49.4°$.

Problem 2: The first calculation result is saved as the variable A.



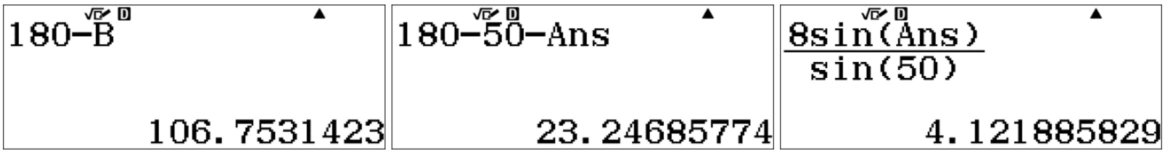
$\angle A \approx 55.8°$, $\angle B \approx 41.4°$ and $\angle C \approx 82.8°$.

Problem 3, solution 1:



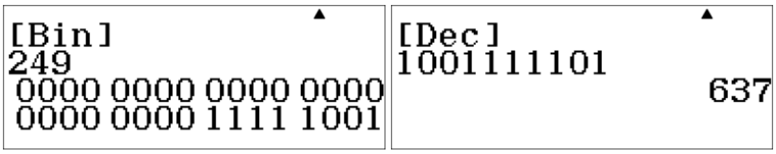
$\angle B \approx 73.2°$, $\angle C \approx 56.8°$ and $c \approx 8.73$.

Problem 3, solution 2:

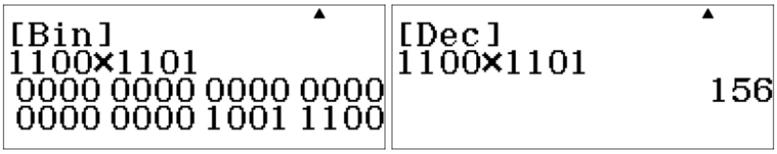


$\angle B \approx 106.8°$, $\angle C \approx 23.2°$ and $c \approx 4.12$.

Problems 4 and 5:



Problem 6:



Problem 7:

Ann's total time:

$$\frac{200}{70} \qquad 2°51'25.71"$$

Peter's total time:

$$\frac{100}{80}+\frac{100}{60} \qquad 2°55'0"$$

Ann is back at $11:51.26$ and Peter returns at $11:55.00$. On the way back, Peter and Ann have driven the same distance after, let's say, x hours. In x hours, Ann drives $70x$ km and Peter drives $100+60\left(x-\frac{100}{80}\right)$ km. These distances are the same after 2.5 hours:

$$70x=100+60\left(x-\frac{100}{80}\right) \qquad x= 2.5 \qquad L-R= 0$$

$$x \qquad 2°30'0"$$

$$70x-100 \qquad 75$$

On the way back Ann reaches Peter $11:30.00$ at a distance 75 km from the city B.

EPILOGUE

Main responsibility and credit for the content belongs to Bjørn, who compressed his years of expertise and few favorite topics in this book. Some additional material was created by Pepe, who also finalized the lay-out.

Tools used to create graphics in this book is ClassPad.net and Casio EDU+ service, all screen shots are from the ClassWiz Emulator.

We hope you find this book helpful during your journey towards the beauty of mathematics.

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CLASSWIZ
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AND SCIENCE FUN